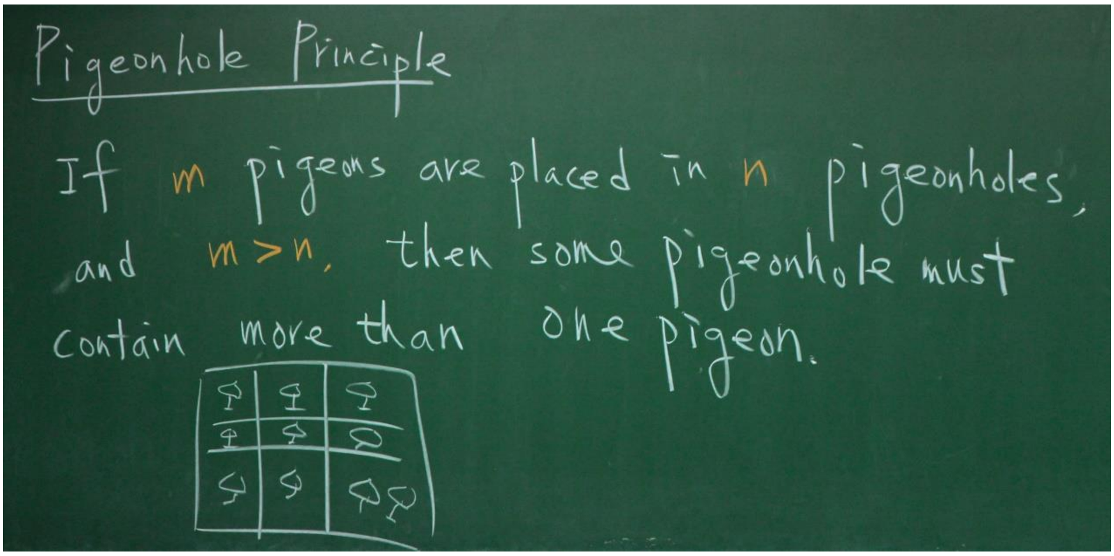
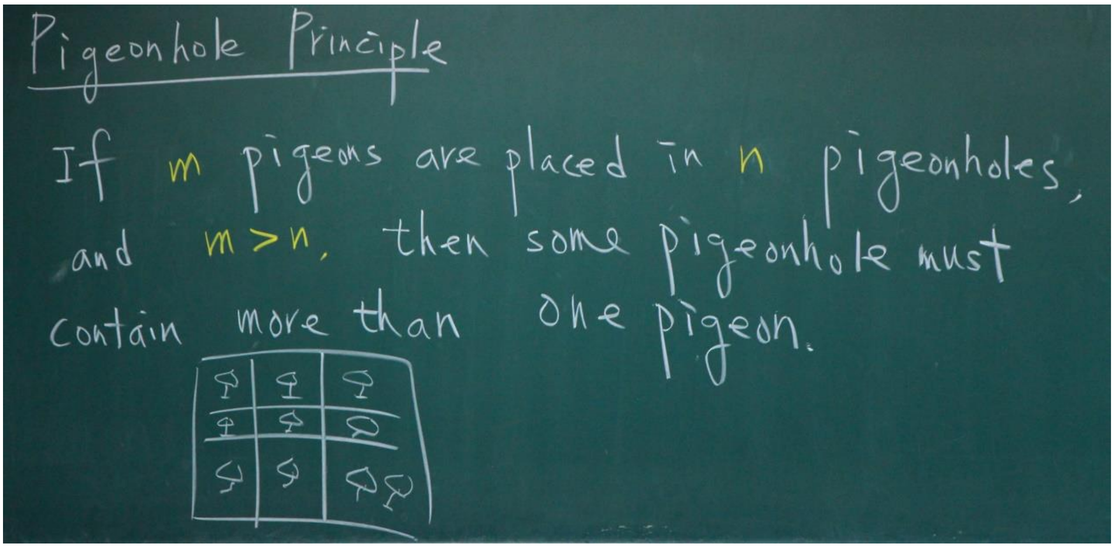
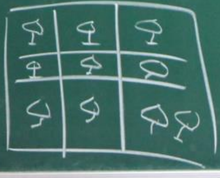
【10920 趙啟超教授離散數學 / 第 5 堂版書】





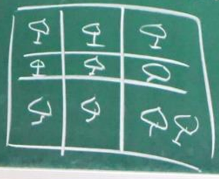
and $m > n$, then some pigeonhole must contain more than one pigeon.



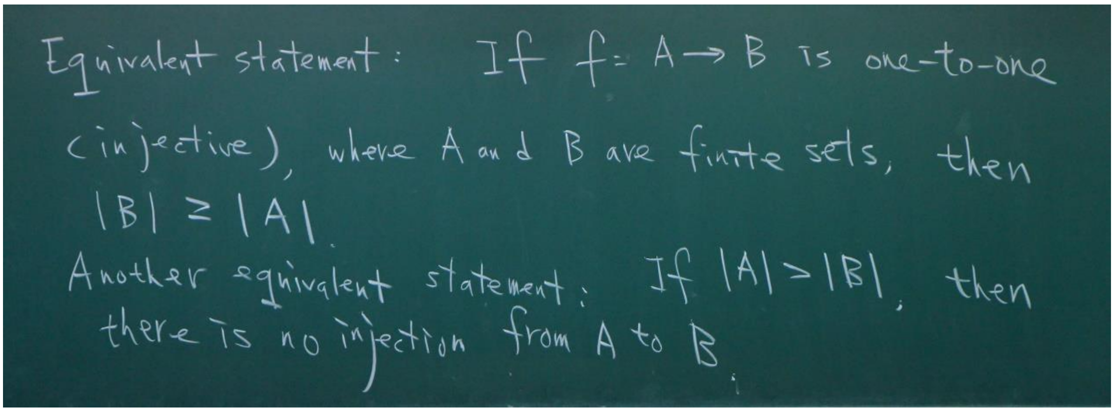
Proof If there is at most one pigeon in each pigeonhole, then there are at most n pigeons totally, contradicting the assumption that there are m pigeons and $m > n$. ∎

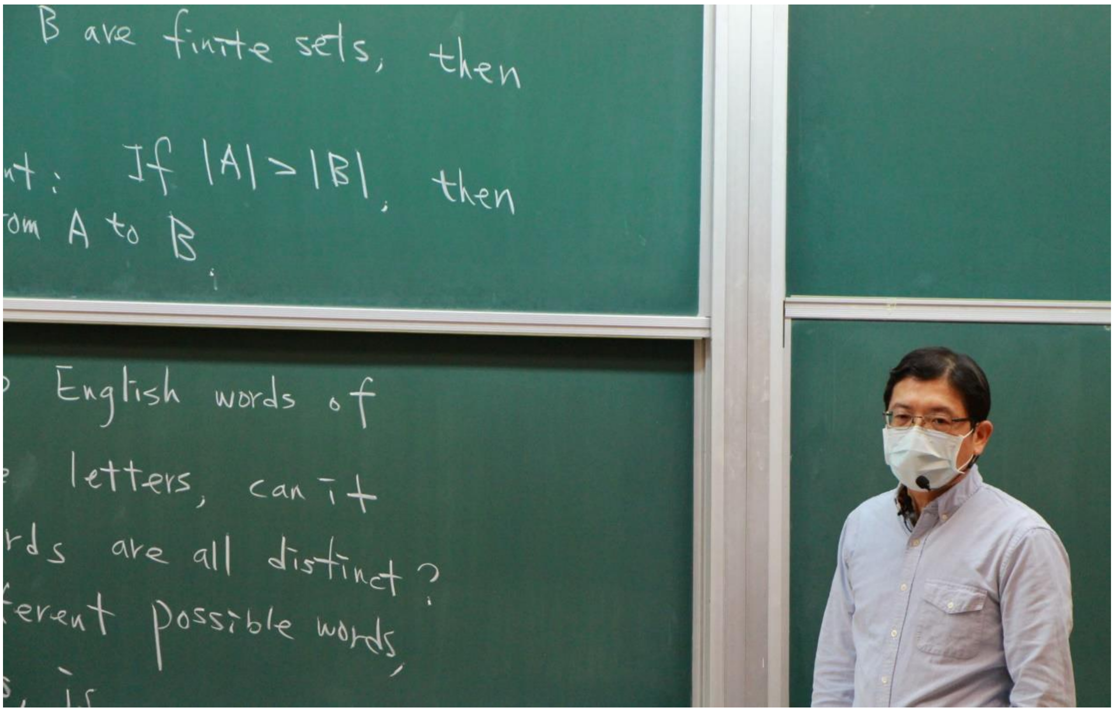
Pigeonhole Principle

If m pigeons are placed in n pigeonholes,
and $m > n$, then some pigeonhole must contain more than one pigeon.



Proof If there is at most one pigeon in each pigeonhole, then there are at most n pigeons totally, contradicting the assumption that there are m pigeons and $m > n$. ∎


Equivalent statement: If $f: A \to B$ is one-to-one
(injective), where A and B are finite sets, then
$|B| \geq |A|$.
Another equivalent statement: If $|A| > |B|$, then
there is no injection from A to B.


B are finite sets, then
nt: If $|A| > |B|$, then
om A to B.
English words of
letters, can it
rds are all distinct?
ferent possible words,

(injective), where A and B are finite sets, then $|B| \geq |A|$.

Another equivalent statement: If $|A| > |B|$, then there is no injection from A to B.

<u>Example</u> Given 500,000 English words of four or fewer lower case letters, can it be that the 500,000 words are all distinct?

The total number of different possible words, with four or fewer letters, is

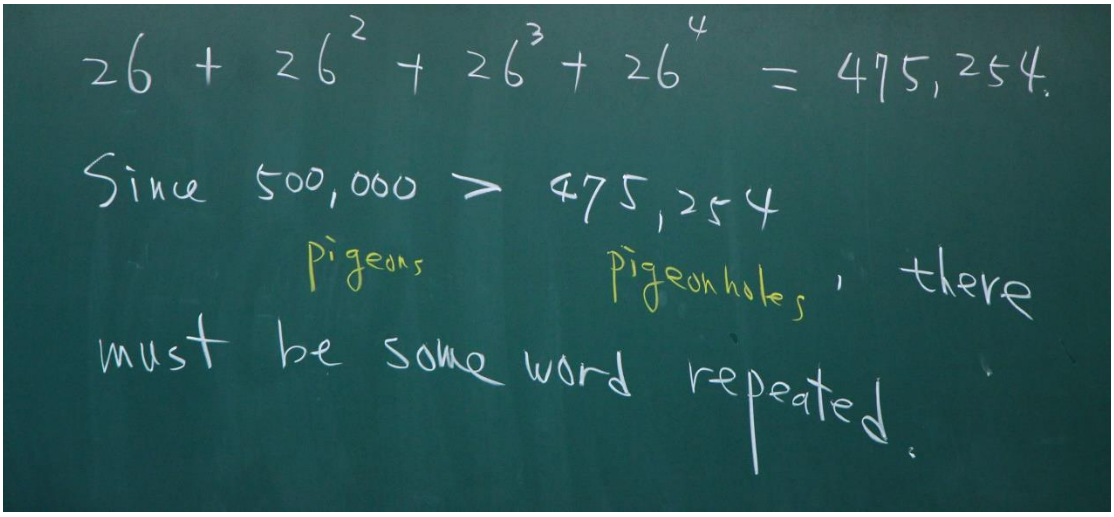
Equivalent statement: If $f: A \to B$ is one-to-one

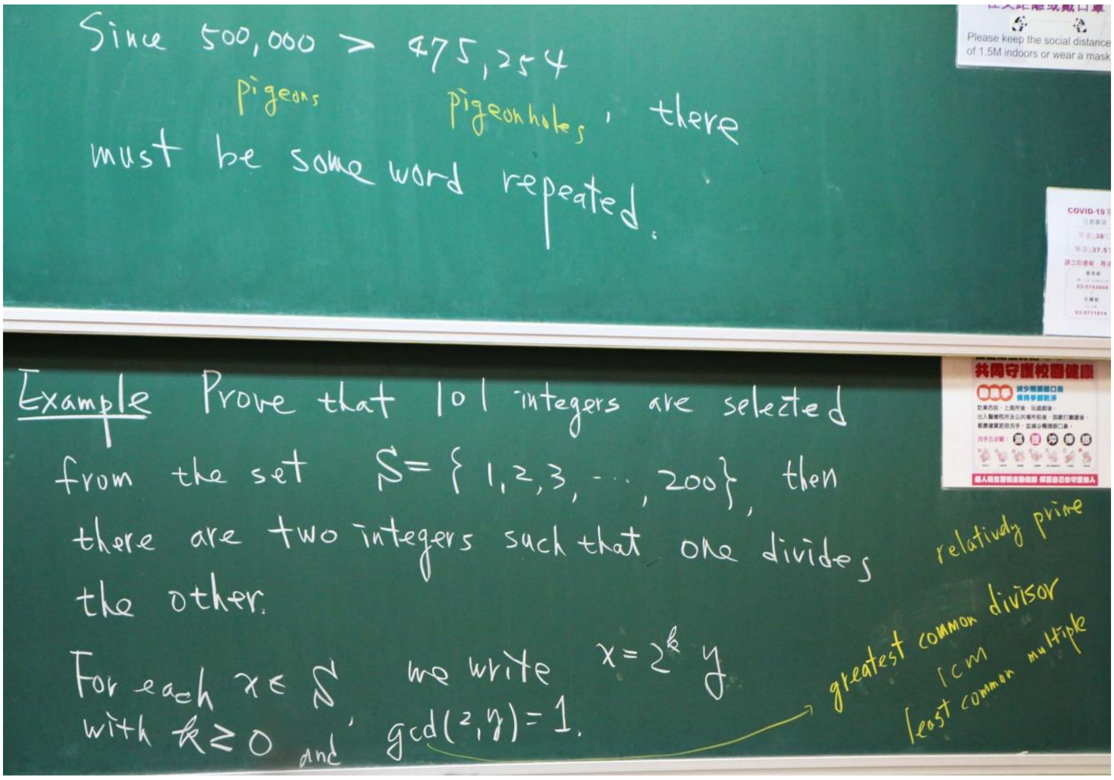
(injective), where A and B are finite sets, then $|B| \geq |A|$.

Another equivalent statement: If $|A| > |B|$, then there is no injection from A to B.

<u>Example</u> Given 500,000 English words of four or fewer lower case letters, can it be that the 500,000 words are all distinct?

The total number of different possible words, with four or fewer letters, is


$26 + 26^2 + 26^3 + 26^4 = 475,254$.
Since 500,000 > 475,254
pigeons
pigeonholes, there
must be some word repeated.


Please keep the social distance of 1.5M indoors or wear a mask
Since 500,000 > 475,254
pigeons
pigeonholes, there
must be some word repeated.
COVID-19
Example Prove that 101 integers are selected
from the set $S = \{1, 2, 3, \cdots, 200\}$, then
there are two integers such that one divides
the other.
relatively prime
greatest common divisor
lcm
least common multiple
For each $x \in S$ we write $x = 2^k y$
with $k \geq 0$ and $\gcd(2, y) = 1$.

$26 + 26^2 + 26^3 + 26^4 = 475,254$.

Since $500,000 > 475,254$
pigeons pigeonholes, there
must be some word repeated.

Example Prove that 101 integers are selected from the set $S = \{1, 2, 3, \cdots, 200\}$, then there are two integers such that one divides the other.

For each $x \in S$, we write $x = 2^k y$
with $k \geq 0$ and $\gcd(2, y) = 1$.

relatively prime
greatest common divisor
lcm
least common multiple

Then y is odd, so $y \in T = \{1, 3, 5, \cdots, 199\}$.
where $|T| = 100$.

Since 101 integers are selected from S, by the pigeonhole principle, there are two distinct integers of the form $a = 2^m y$, $b = 2^n y$ for some (the same) $y \in T$.

If $m < n$, $a \mid b$; otherwise, we have
$m > n$ and $b \mid a$.

(For $a, b \in \mathbb{N}$, $a \mid b \Leftrightarrow b = a \cdot c$ for some $c \in \mathbb{N}$.)
"a divides b"
"a is a divisor of b"
"b is a multiple of a"

Then y is odd, so $y \in T = \{1, 3, 5, \cdots, 199\}$.
where $|T| = 100$.

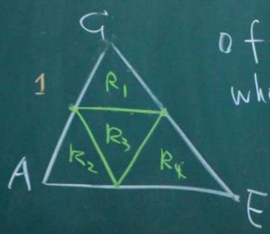
Since 101 integers are selected from S, by the pigeonhole principle, there are two distinct integers of the form
$a = 2^m y$, $b = 2^n y$ for some (the same) $y \in T$.

$2^2 \cdot 7$ $\quad$ $2^3 \cdot 7$

If $m < n$, $a \mid b$; otherwise, we have
$m > n$ and $b \mid a$.

(For $a, b \in \mathbb{N}$, $a \mid b \iff b = a \cdot c$ for some $c \in \mathbb{N}$.)
"a divides b"
"a is a divisor of b"
"b is a multiple of a."

<u>Example</u> Triangle ACE is equilateral with $\overline{AC} = 1$.
If five points are selected from the interior of the triangle, there at least two whose distance apart is less than $1/2$.



Five points in the interior of triangle ACE must be such that at least two of them are

Five points are selected from the interior of the triangle. there at least two whose distance apart is less than $1/2$

Five points in the interior of triangle AGE must be such that at least two of them are in one of the four regions R_i, $1 \le i \le 4$, where any two points are separated by a distance less than $1/2$.

<u>Example</u> Show that in any finite set A of people, there are two members of A who have the same number of friends in A. (It is assumed that $|A|$ is at least 2 and if x is a friend of x' then x' is a friend of x.)

of people, there are two members of A
who have the same number of friends in A.
(It is assumed that $|A|$ is at least 2
and if x is a friend of x' then
x' is a friend of x.)

Define the function f, for each member x
of A, by $f(x) =$ the number of friends
of x in A. If $|A| = m$, then f is a
function from A to the set $B = \{0, 1, 2, \cdots, m-1\}$

<u>Example</u> Show that in any finite set A

of people, there are two members of A
who have the same number of friends in A.
(It is assumed that $|A|$ is at least 2
and if x is a friend of x' then
x' is a friend of x.)

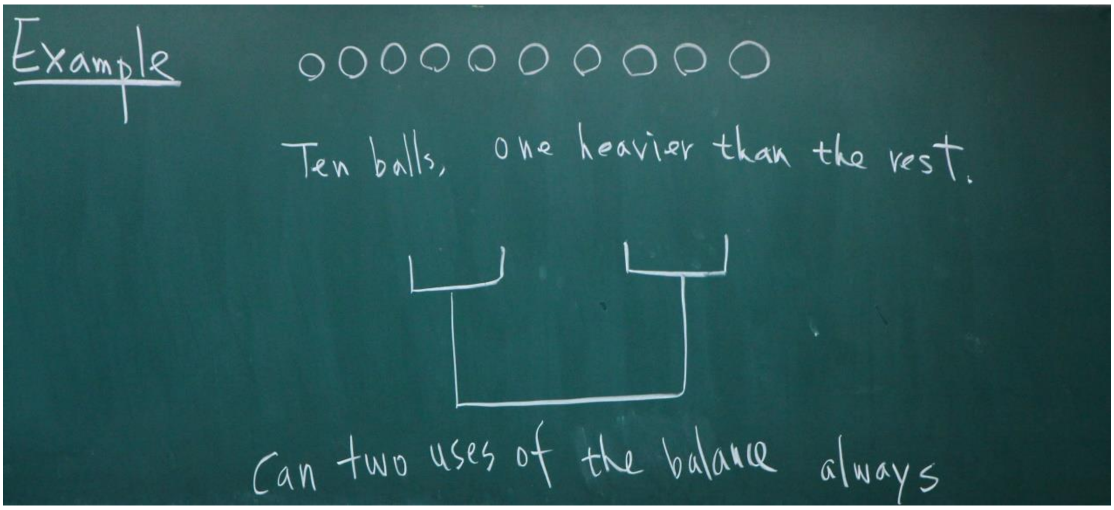
Define the function f, for each member x
of A, by $f(x) =$ the number of friends
of x in A. If $|A| = m$, then f is a
function from A to the set $B = \{0, 1, 2, \cdots, m-1\}$

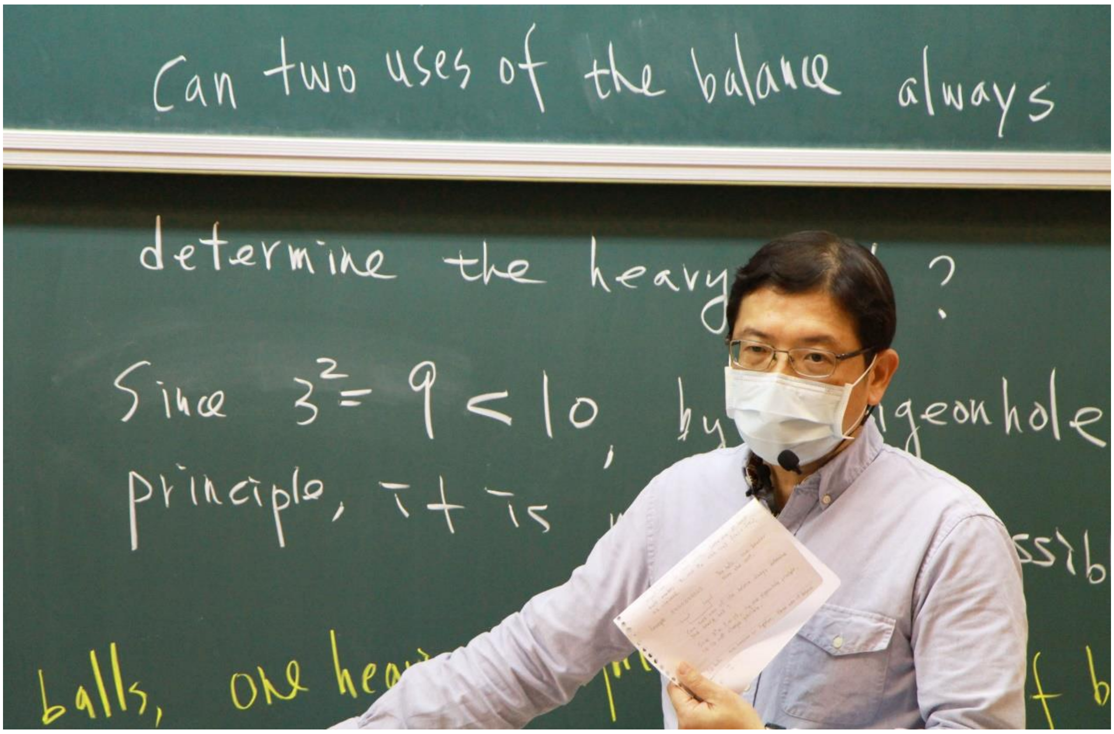
However, if a person x^* has $m-1$ friends, then everyone else is a friend of x^*, and consequently no one has no friends. In other words, the numbers of $m-1$ and 0 cannot both be the values of f. Hence f is a function from a set with m members to a set with at most $m-1$ values,

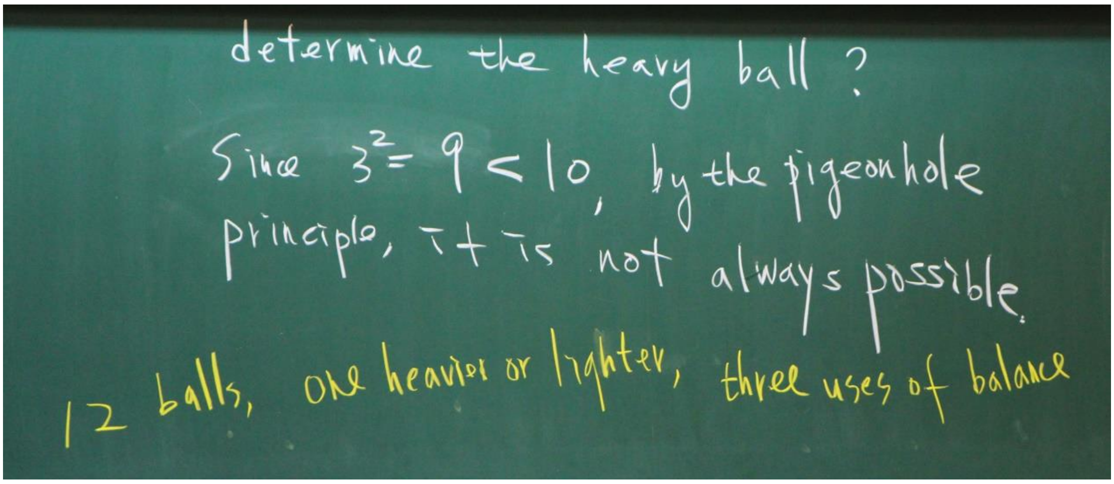
and the pigeonhole principle tells us that f is not injective. In other words, there are at least two members x_1 and x_2 such that $f(x_1)=f(x_2)$, as claimed.

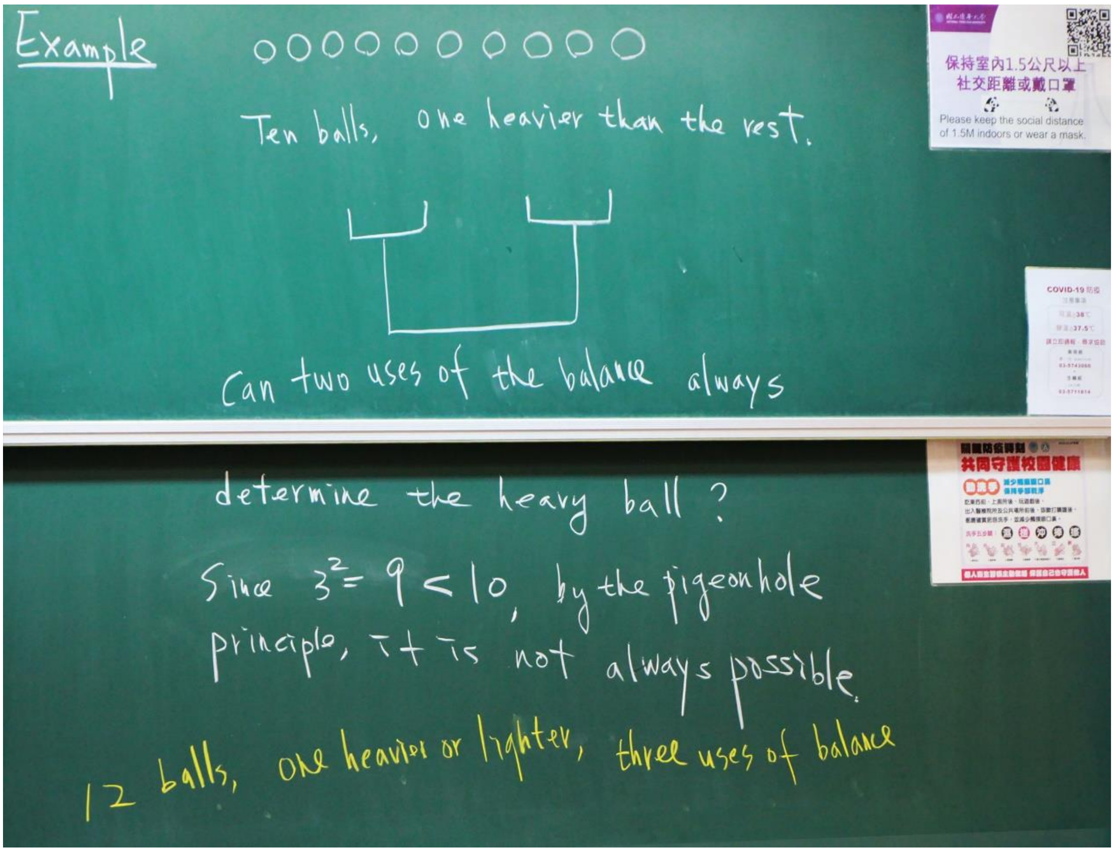
However, if a person x^* has $m-1$ friends, then everyone else is a friend of x^*, and consequently no one has no friends. In other words, the numbers of $m-1$ and 0 cannot both be the values of f. Hence f is a function from a set with m members to a set with at most $m-1$ values,

and the pigeonhole principle tells us that f is not injective. In other words, there are at least two members x_1 and x_2 such that $f(x_1)=f(x_2)$, as claimed.


Example
Ten balls, one heavier than the rest.
Can two uses of the balance always


Can two uses of the balance always
determine the heavy
Since $3^2 = 9 < 10$, by
geonhole
principle, it is
balls, one


determine the heavy ball ?
Since $3^2 = 9 < 10$, by the pigeonhole principle, it is not always possible.
12 balls, one heavier or lighter, three uses of balance


Example
Ten balls, one heavier than the rest.
Can two uses of the balance always
determine the heavy ball ?
Since $3^2 = 9 < 10$, by the pigeonhole principle, it is not always possible.
12 balls, one heavier or lighter, three uses of balance
保持室內1.5公尺以上
社交距離或戴口罩
Please keep the social distance of 1.5M indoors or wear a mask.
共同守護校園健康

of A, by $f(x) =$ the number of friends of x in A. If $|A| = m$, then f is a function from A to the set $B = \{0, 1, 2, \cdots, m-1\}$.

Let A and B finite sets.

If $f: A \to B$ is onto (surjective), then $|A| \geq |B|$.

If $f: A \to B$ is a one-to-one correspondence (bijective) then $|A| = |B|$.